Radiation from a Uniformly Accelerated Charge and the Equivalence Principle

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Abstract

The question of whether a uniformly accelerating charge emits radiation or not is still an open question in electromagnetism. This paper treats the charge classically while quantizing the photon field and using the Feynman propagator to present a new perspective on this age-old problem. The result is a nonzero probability of detecting radiation. This paper's interpretation is that, from the charge's perspective, a photon pair is created due to the Unruh effect and that both the charge and the detector see one photon as a result. This paper shows how radiation from the Unruh effect validates the equivalence principle even for charged particles.

Contents

1 Introduction

The complete and elegant formulation of electromagnetism through Maxwell's laws once led physicists of the likes of Lord Kelvin to claim in 1900 that "There is nothing new to be discovered in physics now. All that remains is more and more precise measurement" [1]. In light of this bold statement (spoken just before the advent of quantum mechanics and general relativity), it might seem surprising that any fundamental discrepancies remain in the theory of electromagnetism. These do, in fact, exist, and one of them is the subject of this paper. The question can be posed simply: Does a uniformly accelerated charge radiate?

Of course, this may at first seem a silly question for those who know of bremsstrahlung and synchrotron/cyclotron radiation...of course an accelerating charge radiates! A typical undergraduate electromagnetism class introduces the famed Larmor formula, which can be adapted to relativistic concerns (via Liénard's generalization) and takes the form

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right) \tag{1}$$

where P is the radiated power, q is the charge's electromagnetic charge, γ is the Lorentz factor from relativity, and v and a are the charge's velocity and acceleration, respectively [2]. This is clearly nonzero for nonzero (linear) acceleration.

So why is there a question as to whether or not a uniformly accelerating charge radiates? Feynman suggests that "we have inherited a prejudice that an accelerating charge should radiate" [3], and that if we examine the assumptions of these traditional equations, the answer is far less clear-cut. For example, Equation 1 is only applicable to periodic or bounded motion. Radiation also appears to pose problems for special relativity: if an accelerating charge radiates but one sitting in a lab on Earth in a gravitational field does not, then is the equivalence principle wrong? Does it not apply to the global phenomenon of charged particles?

This paper forms a cohesive version of how radiation and the equivalence principle fit together; indeed, radiation will turn out to be a crucial element of the final interpretation. After discussing previous methods and background information in Sections 2 and 3, respectively, Section 4 presents an original calculation, combining classical and quantum techniques to determine the probability that a two-level Minkowski detector will detect radiation (i.e. transition to its excited state) as it watches a classical charge with constant proper acceleration. In this way, we provide a new perspective on the "perpetual problem" of radiation of an accelerating charge.

2 Previous Methods and Subtleties

This section explores previous methods of resolving whether or not an accelerating charge radiates. Once the method is presented, potential flaws and/or subtleties in the calculations are revealed that demonstrate why the new and different approach of this paper is necessary.

2.a Liénard-Wiechert Potentials and the Poynting Vector Flux

One of the most traditional methods of calculating radiation involves finding the flux of the charge's Poynting vector at infinity. This is the approach followed by Griffiths in his electromagnetism textbook [2]. The method is conceptually straightforward: find the scalar and vector potentials of a point charge moving at relativistic speeds (thereby accounting for the finite speed of light), from which one can, after some intense and tedious calculation, derive the electric and magnetic fields of the charge and thereby the Poynting vector.

The process of deriving the fields of a point charge, however, has a long history fraught with causality violations and misinterpretations. The first to attempt such a feat was Born, who did not bother with radiation at the time [4]. Schott independently calculated the fields in 1912 and subsequently claimed that the charge would radiate [5, 6]. At least two physicists use Born's result to claim that a charge would not radiate: Pauli in 1921 and von Laue in 1911 [7, 8]. Also in 1921, however, Milner found that Born's fields violated causality, in that the calculated field was nonzero everywhere, whereas, due to the finite speed of field propagation, not everywhere should experience an electric field [9]. Milner proposed that Born's solution was actually two accelerating charges, but was contested by Bondi and Gold in 1955. Bondi and Gold then proposed new fields by adding a delta function to the Born fields and claimed that the charge would radiate [10]. The delta function addition was debated by Fulton and Rohrlich in 1960, who calculated that the charge would radiate at a constant rate [11].

The final result of [11] matches with standard textbooks such as [2] and is

$$
\begin{aligned}
P &= \oint \vec{S}_{\text{rad}} \cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\phi \\
&= \frac{\mu_0 q^2}{6\pi c} a^2
\end{aligned} \tag{2}
$$

This is the Larmor formula (non-relativistic, in SI units) and is clearly nonzero for nonzero acceleration (although this formula is not applicable to uniform non-relativistic accelerations [12]).

However, as Fulton and Rohrlich first suggest, there may be an important distinction between accelerating for all time and accelerating for a finite but arbitrarily long time [11].

A similar argument is posed years later in 1997 by Parrott, likening the difference in finite and infinite time to the need for careful treatment of infinite sums that give entirely different results when summed in different orders [13]. Indeed, Parrott uses an earlier 1997 paper's argument which yields no radiation when applied to infinite time [14] to show the existence of radiation for a finite time [13].

Clearly, even when the calculation of the fields has been agreed upon, there are still subtleties in how to define the problem. Indeed, some argue that defining fields at infinity has issues when considering retarded times [15], and Feynman also states that the Poynting vector method is no longer valid for continuously growing motion [3]. Already, this purely classical calculation looks convoluted to interpret.

2.b Radiation Reaction Force and Feynman's Approach

The classical method often relies on interpreting the "radiation pressure," or "reaction force" experienced by the charge. The Abraham-Lorentz formula blames the loss of energy to radiation on the reaction force, expressed as

$$\vec{F}_{\text{rad}} = \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{a}} \tag{3}$$

as illustrated in [2] (note that Gaussian units are employed here). The Abraham-Lorentz formula (3) clearly says that the radiation reaction force is equal to zero in the case of a point charge that is uniformly accelerating.

Feynman also uses this equation to justify why an accelerating charge might not radiate. He begins with the Abraham-Lorentz formula (3) and derives the work done against this force:

$$\frac{dW}{dt} = -\frac{2}{3}\frac{e^2}{c^3}\vec{v}\cdot\dot{\vec{a}} \tag{4}$$

which can be re-written as

$$\frac{dW}{dt} = \frac{2}{3}\frac{e^2}{c^3}\vec{a}\cdot\vec{a} - \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\left(\vec{v}\cdot\vec{a}\right) \tag{5}$$

The first term is the Larmor formula and the second drops out for periodic motion. Feynman claims that the true form is illustrated in Eq. 4 and thus constant acceleration does not yield radiation [3, Lecture 9].

However, the validity of the Abraham-Lorentz equation is questionable. Indeed, Eq. 3 results in either strange "runaway" behavior in which the charge acceleration increases exponentially over time or acausal solutions in which the charge accelerates before the force even acts [16] (see [17] or [2, Ch. 11] for a further discussion). As hard as it is to accept such an equation, a hefty part of the literature concerning this problem involves making sense of the radiation reaction or trying to get rid of it.

Curiously, even the presence or absence of radiation reaction seems not to solve the question of whether the charge radiates or not. From Feynman's point of view, no radiation reaction means no radiation [3, 18]. Fulton and Rohrlich, however, suggest that radiation exists even though the radiation reaction is zero [11]. They argue that considering the "acceleration energy", whose time derivative vanishes for bounded or periodic motion, leads to the idea that no radiation reaction means no radiation. However, they point out that since uniform acceleration is neither periodic nor bounded, it is possible to have radiation with no radiation reaction. This view is also taken up by more recent papers, which claim that the near field of the charge itself is redistributed and thus allows for radiation without radiation reaction [19, 16].

Another important view on the radiation reaction problem is suggested by Boulware in 1980. According to him, the radiation reaction vanishes. Interestingly, he argues that the charge radiates even when viewed by a co-accelerating observer, but the observer simply cannot see the radiation due to the horizons of the space-time [20]. He claims that it is not possible to separate out the $1/r$ decay of the radiation fields within the co-accelerating observer's field of view: this argument will become especially important when discussing the implications for the equivalence principle in Section 2.c. Parrott turns this upside down by arguing that Boulware incorrectly identified the conserved quantity of energy that does or does not produce radiation, and that there probably is radiation reaction [21]. However, Parrott suggests that the radiation energy comes entirely from the energy that pushes the charge into and out of uniform acceleration in the first place; that is, that produces $\dot{\vec{a}}$, and in this way attempts half-heartedly to justify the Abraham-Lorentz equation of Section 2.a [21].

As a side note, Feynman appears to have changed his mind on this problem: a 1945 paper with Wheeler tries to derive the radiation due to the reaction force via absorbers excited in the future that then emit backward-propagating waves [22].

The past two sections have concentrated on outlining details of previous calculations; placing these results in the framework of special relativity makes the situation even more complicated.

2.c The Equivalence Principle

Perhaps the most intriguing aspect of this problem is its implication for the equivalence principle. Take the following anecdote from Parrott [21]: imagine that a charged particle is accelerating in flat spacetime by means of a rocket, which contains some trackable amount of fuel. If the charged particle radiates, then the rocket must burn more fuel to keep the same acceleration as compared to an uncharged particle (a view also stated in [2]). Now consider the charge in a uniform gravitational field, which is equivalent from the charge's perspective to accelerating in flat spacetime. Here, a co-accelerating observer sees no radiation and thus rockets propelling charged and uncharged particles would expend the same amount of fuel. But the whole idea of the equivalence principle is that these two situations should be indistinguishable. Clearly, if we can just measure the amount of fuel that the two rockets use up and compare then, we would be able to tell (locally) the difference between accelerating in flat spacetime and standing still in a constant gravitational field. What's going on? Is the equivalence principle being violated?

It is worth noting that the equivalence principle is supposed to be valid only locally, whereas one can always tell globally whether or not a charge is present, since its electric field will always be present, no matter how far away the observer is. In this way, one might say that a charge is not a local phenomenon, and thus we have no reason to expect that the equivalence principle would even apply. This is the view taken by Parrott even as he shows that Boulware's validation of the equivalence principle is wrong [21]. For many, however, the equivalence principle is worth saving.

It is also somewhat surprising that the equivalence principle has been misapplied by many, mislead by the idea that radiation should be observer-invariant. The observer and who is registering the radiation must be very clearly defined from the outset. For example, at least one paper equated a stationary Minkowski observer watching an accelerating charge to an observer stationary in a gravitational field watching a stationary charge [15] (early papers thus explicitly state that the two are not related by any Lorentz transformation [23]). Another paper claims that radiation must be absolute because the number of photons measured for different observers must be the same [24]. However, this is definitely not the case in the quantum world (for example, the Schwinger effect allows for electron-positron pairs to be created when the potential energy of the electric field exceeds their rest energy [25]). This lack of particle number conservation is one motivation for the present paper's calculation to employ quantum mechanical techniques rather than purely classical ones. Rohrlich seems to have sorted this out and is the first to suggest that radiation is not an absolute concept [23].

Of course, there is no problem of contradicting the equivalence principle if an accelerating charge does not radiate. In this case, a charged and uncharged particle would have the same worldline and the same amount of rocket fuel would be used. This is arguably why Feynman and Singal both claim that there is no radiation [3, 14]. With more evidence mounting against this view, however, other methods of justifying the equivalence principle have arisen.

There are a few cop-out explanations that attempt to rescue the equivalence principle. For example, Bondi and Gold claim that the equivalence principle could theoretically be violated, but would never be violated in nature since the necessary gravitational field would be unnatural [10]. However, the Schwarzschild solution of the vacuum Einstein field equations does just that. Fulton and Rohrlich suggest that radiation cannot be detected locally and hence even though the two situations are not physically equivalent, they are locally equivalent [11]. Boulware justifies this rigorously by showing that radiation would be behind a horizon [20]. This result is also derived in a more intuitive format in de Almeida's 2005 paper, although this paper assumes off the bat that the charge radiates [26]. Again, Parrott claims that these papers have misinterpreted the conserved "energy" [21]. However, arguing that the charge radiates in both situations and that the radiation is simply behind a co-accelerating observer's horizon is a rather appealing way to "save" the equivalence principle.

When the calculations of the fields and radiation reaction and making sense of the equivalence principle is said and done, subtleties still arise. For example, what does it actually mean to "radiate"? The papers above consider radiation to be energy carried off to infinity, but in a quantum setting, one would define radiation as the ping of a detector: if the detector registers a change, radiation has been found. This paper avoids most of the complications of the classical computations in Sections 2.a and 2.b and instead makes use of quantum techniques (outlined in Section 3.b) to shed light on an effect that depends intimately on the quantum nature of the vacuum (as demonstrated in Section 2.d).

2.d The Charge as a Two-level system and the Unruh Effect

The most similar approach to the one that will be taken in this paper is the one used by Unruh and Wald in their 1984 paper [27]. Their paper also uses a two-level system to register radiation; however, their observer is the accelerating object. There is no Minkowski observer; rather they use the state of the spacetime to draw conclusions. In addition, their "radiation" is not photons but rather excitations of a scalar field. This treatment results in unambiguous detection of radiation (although the resolution was debated for a while: see [28, 29, 30]). The present paper essentially seeks to extend the results of Unruh and Wald by asking the same question but with a detector and photons, a vector field, instead of a scalar field. The treatment of the charge as classical rather than a quantum-mechanical two-level system is a simplification of this problem.

The Unruh and Wald paper also reveals the connection of this problem to the Unruh effect: indeed, this will be helpful in the interpretation of Section 4's result. The Unruh effect, as outlined in Unruh's earlier paper [31], is a result of the essential quantum nature of the vacuum. As Carroll points out, there is no reason to expect that the number of particles will be conserved in the relativistic regime, since a particle with enough energy can turn into two particles, etc. [32]. Thereby, an observer accelerating in a Minkowski vacuum actually sees a thermal bath of particles with temperature $kT = \hbar a/2\pi c$ (hence Unruh and Wald's paper explaining what happens when the observer interacts with the bath, i.e., the Minkowski state expected energy of the scalar field increases) [27]. This revelation links the problem of a uniformly accelerated charge to that of Hawking radiation, although the exact role of this effect will be discussed further in Section 6.

Overall, the problem that is the focus of the present paper is intricately linked to the definition of a vacuum, which turns out not to be so simple non-classically. We therefore seek to utilize the propagators of quantum field theory (explained in Section 3.b) by introducing a quantum two-level detector, whose state (excited or ground) will indicate the detection or lack of radiation from an accelerating classical charge. The advantage of this treatment is the avoidance of the subtleties of the Larmor and Abraham-Lorentz formulae, the clear definition of radiation, and the quantum-mechanical perspective on the vacuum.

3 Background Information

The information necessary to understand the calculation in Section 4 is outlined in the sections below. From here on, $c = 1$ and $\hbar = 1$.

3.a Uniform Acceleration (Rindler Coordinates)

In this section, we derive the trajectory of an object with constant proper acceleration in the x-direction as seen by a Minkowski observer. This section is modeled after the discussion in Carroll [32], with the notable difference that the signature is reversed to match that of Peskin & Schroeder, whose signature is also followed in, for example, Section 3.b on propagators [33]: $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Such a uniformly-accelerating observer can be parameterized with respect to the proper time τ of the observer by

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)$$
$$x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau)$$

Taking the derivative twice with respect to τ yields the proper acceleration:

$$a^0(\tau) = \alpha\sinh(\alpha\tau)$$
$$a^1(\tau) = \alpha\cosh(\alpha\tau)$$

This acceleration vector has constant time-like norm $a^\mu a_\mu = -\alpha^2$. The trajectory obeys the formula $t^2(\tau) = x^2(\tau) - 1/\alpha^2$, a hyperboloid with asymptotes at $x = \pm t$. The object's speed increases ever closer to c, but of course never actually reaches it.

We now introduce new coordinates that are adapted to the paths of uniform acceleration. These coordinates (η, ξ) are the Rindler coordinates and are given by the transformation

$$\begin{aligned} t(\eta,\xi) &= \frac{1}{a} e^{a\xi} \sinh(a\eta) \\ x(\eta,\xi) &= \frac{1}{a} e^{a\xi} \cosh(a\eta) \end{aligned} \tag{6}$$

Note that this is only valid for region I, as labeled in Figure 1, which is the fundamental reason why the Minkowski and the Rindler vacuum are not equivalent. Paths along constant ξ are paths of constant acceleration. We set $\xi = 0$ to achieve $\alpha = a$. The metric in this Rindler space is given by

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) + dy^2 + dz^2 \tag{7}$$

The equivalence principle is manifest visually in Figure 1: constant acceleration paths in Rindler coordinates are related to observers sitting at a constant radius from a Schwarzschild black hole (a constant gravitational field). The changes in proper time are changes in velocity, while the time translation symmetry corresponds to the symmetry under Lorentz transformations of Minkowski space [20]. We also note that an observer in region I can receive signals only from region IV and can send but not receive signals from region II. Indeed, this plays a fundamental role in Boulware's argument that an observer in region I cannot see radiation of a co-accelerating charge [20]. According to Boulware, the radiation field at any point is due to the Coulomb potential of the charge plus either the outgoing radiation field of the charge or the incoming radiation field of the charge. In other words, the radiation is either emitted by or absorbed by the charge, and hence a co-accelerating observer will not see any radiation [20].

The origin is a funny place in Rindler coordinates. As Figure 1 shows, constant time slices all run through the origin: thus the origin appears the same at all proper times. It is also the place where all the signals from the Rindler particle's infinite past converge: an observer at the spatial origin would see nothing until $t = 0$, at which point everything from the past would hit almost simultaneously. For this reason (among others to be discussed later), we choose to place our detector at an arbitrary location in region I that is not the origin.

In the present paper we will be interested in a classical charge q with constant proper acceleration and the current it produces. The 4-current in the charge's rest frame is given by

$$J^\mu = (q\delta(\xi)\delta(y)\delta(z), 0, 0, 0)$$

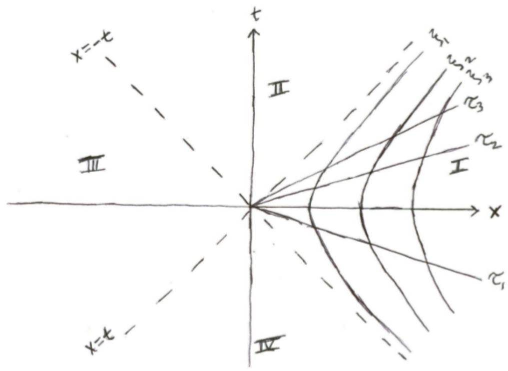

Figure 1: Diagram of the Rindler spacetime. Constant time slices are labeled by τ: here, $\tau_1 < \tau_2 < \tau_3$. Constant accelerations are labeled with ξ: $\xi_1 < \xi_2 < \xi_3$. The horizons are indicated by the dotted lines at $x = \pm t$.

Transforming to Minkowski coordinates:

$$J^{\mu'} = \frac{dx^{\mu'}}{dx^{\mu}} J^{\mu} = \frac{dx^{\mu'}}{d\eta} J^{\eta} = q\delta(\xi)\delta(y)\delta(z) \left[\frac{dt}{d\eta}, \frac{dx}{d\eta}, 0, 0\right]$$
$$= qe^{a\xi}\delta\left(\frac{\ln(a^2(-t^2+x^2))}{2a}\right)\delta(y)\delta(z)\left[\sinh(a\eta), \cosh(a\eta), 0, 0\right]$$

We discard the negative root in the delta function identity $\delta[g(x)] = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$ (where x_i is a root of $g(x)$) because it is always zero in the region we are considering. Calling $x_0 = \frac{1}{a}\sqrt{1+t^2a^2}$, and plugging in $\alpha = a$ and thus $\xi = 0$ and $\eta = \tau$, the proper time of the particle, the 4-current as seen in the Minkowski frame becomes

$$J^{\mu'}(t(\tau)) = \frac{q}{x_0}\delta(x - x_0)\delta(y)\delta(z)\,(x, t, 0, 0) = q\delta(x - x(\tau))\delta(y)\delta(z)v^{\mu'} \tag{8}$$

This expression makes sense: the charge density is still q and, remembering that $I = \rho v = \rho\frac{dx}{dt} = \rho\frac{dx}{d\tau}\frac{d\tau}{dt} = q\sinh(a\tau)\frac{1}{\cosh(a\tau)}$, the current makes sense as well. The far right side, with $v^{\nu} = (1, \tanh(a\tau), 0, 0)$, is a condensation of notation that will be useful in Section 4.

3.b Propagators and Green's Functions

In quantum field theory, the probability for a particle to propagate from position y^{μ} to x_{μ} is given by that particle's so-called propagator, labeled $D(x-y)$. It is helpful to first consider the case of a scalar field, even though this paper will be concerned with photons (a vector field). This section draws from Chapter 2.4 in Peskin & Schroeder [33].

In the Heisenberg picture, the amplitude to propagate from a position y to x is $\langle 0|\phi(x)\phi(y)|0\rangle$ for the scalar field ϕ. The explicit form of the scalar field is

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$$

where $p^0 = E_p$, p^μ is the momentum, and a_p, $a_p^\dagger$ are the creation and annihilation operators for the p mode. This form comes from quantizing the classical theory of a Klein-Gordon field and treating each mode as a harmonic oscillator [33, Chapter 2.3]. It will not be derived in further detail since the scalar field is a stepping stone to the section's ultimate goal of presenting a specific type of propagator (the Feynman and then the photon propagator).

Given the explicit form of ϕ, it is possible to calculate $\langle 0|\phi(x)\phi(y)|0\rangle$:

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

Now, to be consistent, we would like for causality to be preserved. The above propagator, however, results in an exponentially vanishing (but nonzero) amplitude for x and y separated by a spacelike interval ($x^0 - y^0 = 0$, $\vec{x} - \vec{y} = \vec{r}$, for example). The real question, however, is if a measurement at one point could affect a measurement at another point if the points are spacelike-separated. Thus if the commutator $[\phi(x), \phi(y)]$ is zero, the measurements do not affect each other and causality is preserved. It turns out that this commutator is given by

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) \tag{9}$$

Again, details can be found in introductory quantum field theory books [33]. Now, Eq. 9 can be written as $\langle 0|[\phi(x), \phi(y)]|0\rangle$, which in turn can be written as an integral over momentum:

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

The integral has two poles at $p^0 = \pm E_p$, where $E_p = +\sqrt{|\vec{p}|^2 + m^2}$, and the way the poles are dealt with is a distinguishing feature of the physics involved. For example, if we choose a mostly semi-circle contour as shown in Figure 2a, closing in the upper half plane for $x^0 < y^0$ to yield 0 and in the lower half plane for $x^0 > y^0$ to pick up both poles, the result is $D_R(x-y) = \theta(x^0 - y^0)\,\langle 0|\phi(x)\phi(y)|0\rangle$. Note that $D_R(x-y)$ is a Green's function for the Klein-Gordon operator; that is, $(\partial^2 + m^2)D_R(x-y) = -i\delta^{(4)}(x-y)$. Since it is nonzero when y^0 happens before x^0, it is the retarded Green's function. Note that this is equivalent to shifting the poles down to lie at $p^0 = \pm E_p - i\epsilon$. Similarly, the advanced Green's function has poles at $p^0 = \pm E_p + i\epsilon$ and is nonzero for $y^0 > x^0$, as shown in Figure 2b. The Feynman

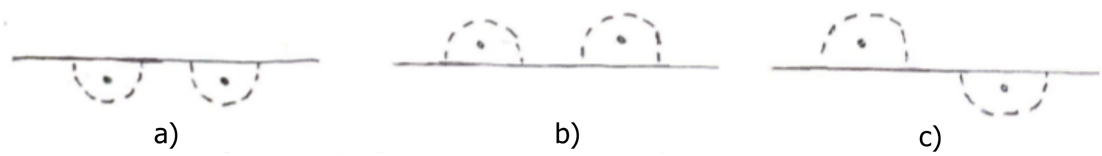

Figure 2: Three methods of dealing with the propagator poles: retarded, advanced, and the Feynman prescription from left to right. The contours are closed depending on the values of t_f and t_0, or, equivalently, the poles are shifted below/above the real axis.

prescription includes only one pole on either side of the contour as seen in Figure 2c: this amounts to poles at $p^0 = \pm(E_p - i\epsilon)$. The full form of the Feynman propagator is

$$\begin{aligned} D_F(x-y) &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p-(E_p - i\epsilon))(p+(E_p-i\epsilon))} e^{-ip\cdot(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \end{aligned} \tag{10}$$

The Feynman propagator is also a Green's function of the Klein-Gordon operator. When $x^0 > y^0$, the integration yields $D(x-y)$, and when $x^0 < y^0$, it gives $D(y-x)$. Thus $D_F(x-y)$ can be written as

$$\begin{aligned} D_F(x-y) &= \theta(x^0 - y^0) \langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0) \langle 0|\phi(x)\phi(y)|0\rangle \\ &= \langle 0|T\phi(x)\phi(y)|0\rangle \,, \end{aligned}$$

where T is the time-ordering symbol, which says to act the operators in the order that they happen.

Now that propagators have been explained and a basic version (Eq. 10) for a scalar field has been derived, we can now approach the propagator that is actually needed for this paper: the photon propagator. The photon propagator is derivable by quantizing the appropriate action's functional integral [33, Chapter 9.4], but here we take it on faith for the purposes of conciseness. This form given below in Eq. 11 is plausible as it is almost identical to the propagator in Eq. 10 (and photons also satisfy the massless Klein-Gordon equation) with the metric tacked on to account for all the intermediate states possible for the vector field.

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{11}$$

A_μ is the vector potential and $\eta_{\mu\nu}$ is the Minkowski metric, signed as in Section 3.a.

3.c State Evolution via Time Dependent Perturbation Theory

It is useful to express the effects of an interaction as a perturbation to the free (non-interacting) theory, since interacting theories can rarely be solved exactly. To this end, write $H = H_0 + H_1$, where the eigenstates of H_0, the unperturbed Hamiltonian, are known, and H_1 is a small perturbation. Going to the interaction picture, the states and operators evolve from time t_0 to time t_f according to the unperturbed Hamiltonian, that is,

$$
\begin{aligned}
H_I &= e^{iH_0(t_f-t_0)} H_1 e^{-iH_0(t_f-t_0)} \\
|\psi_I(t)\rangle &= e^{iH_0(t_f-t_0)} |\psi_S(t)\rangle = e^{iH_0(t_f-t_0)} e^{-iH(t_f-t_0)} |\psi_S(t_0)\rangle = U_I |\psi_I(t_0)\rangle
\end{aligned}
$$

where $|\psi_I\rangle$ is the wave function in the interaction picture and $|\psi_S\rangle$ is in the Schrödinger picture. H_I and U_I are the Hamiltonian and time evolution operator, respectively, in the interaction picture (see [34, Section 14.5]). We see that the time evolution operator satisfies the differential equation:

$$
i\frac{\partial U}{\partial t} = e^{iH_0(t_f-t_0)} H_1 e^{-iH_0(t_f-t_0)} e^{iH_0(t_f-t_0)} e^{-iH(t_f-t_0)} = H_I U_I
$$

which follows since H_0 and $e^{iH_0(t_f-t_0)}$, H and $e^{-iH(t_f-t_0)}$ commute. This equation is solved (given initial condition $U(t_f = t_0, t_0) = 1$), by:

$$
U_I(t_f, t_0) = 1 - i\int_{t_0}^{t_f} dt' H_I(t') U_I(t', t_0)
$$

The form of the explicit solution can be found by first attempting a recursive solution. The result is:

$$
U_I(t_f, t_0) = e^{T\left[-i\int_{t_0}^{t_f} H_I(t')dt'\right]}
$$

The time-ordering symbol allows the expression to simplify a bit since the integral is symmetric about $t' = t''$ and thus counts everything twice (hence the factor of $1/2$). See [33, Section 4.2]. The T just means that the Hamiltonians act in the correct order: a photon isn't destroyed before it's created and so on. This solution is exact: however, for the purposes of this paper, the second-order expansion is sufficient. We therefore take

$$
U_I(t_f, t_0) = 1 - i\int_{t_0}^{t_f} dt' H_I(t') - \frac{1}{2}\int_{t_0}^{t_f} dt' dt'' T[H_I(t') H_I(t'')] \tag{12}
$$

3.d Important Hamiltonians

This paper uses two main Hamiltonians: that of a classical charge, and that of a two-level system, both interacting with the photon field.

3.d.1 Classical Charge

To include the effects of a uniformly accelerating charge, this paper will be concerned with the Hamiltonian for a classical charge and a quantized electromagnetic field. The total Hamiltonian for this system has two components: the first, the free Hamiltonian for a relativistic charged particle, and the second, the interaction between the vector potential (the electromagnetic field) and the charge (which comes in through the charge's current). The free Hamiltonian can be found in, for example, Jackson's electromagnetism book [35, Section 12.5]: $H_{C_{free}} = \sqrt{(c\vec{P} - e\vec{A})^2 + m^2c^4} + e\Phi$. The interacting potential is given by

$$H_{C1} = \int d^3x J^\mu(\vec{x}) A_\mu(\vec{x}) \tag{13}$$

where J^μ and A_μ are the 3-vector current and vector potential, respectively [33, Section 4.8]. This paper treats the charge as classical and the electromagnetic field as a quantized object: hence A_μ here includes the photon creation and annihilation operators and is given by

$$\vec{A}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=0}^{3} \left(a^r_{\mathbf{p}} \epsilon^r_\mu(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + a^{r\dagger}_{\mathbf{p}} \epsilon^{r*}_\mu(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right) \tag{14}$$

also found in [33, Section 4.8]. E_p is the energy of each mode, here equal to the magnitude of $\vec{p}$. r labels a basis for the polarization vectors ϵ_μ that solve, in the Lorentz gauge, the equation $\partial^2 A_\mu = 0$, satisfying $p^\mu \epsilon_\mu = 0$. For a detector located at the origin such that the photons it detects propagate along the x-axis, some possible choices are circularly polarized bases: $\epsilon_\mu = (0, 0, 1, \pm i)$ as well as the simpler choices $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ (note that there are only two: the four-part sum collapses to two components that are meaningful in terms of photons). For a detector not located at the origin, the form is different to ensure that there are two basis vectors perpendicular to the direction of propagation; however, the polarization vectors do not play a significant role in this paper since they will just be absorbed into the photon propagator and as such will not be discussed in detail.

Since the perturbation techniques employed in this paper require that the interaction Hamiltonians be in the interaction picture, we add time-dependence to the vector potential as follows:

$$A_\mu(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1}^{2} \left(a^r_{\mathbf{p}} \epsilon^r_\mu(p^\mu) e^{-ip^\mu x_\mu} + a^{r\dagger}_{\mathbf{p}} \epsilon^{r*}_\mu(p^\mu) e^{ip^\mu x_\mu} \right) \tag{15}$$

and similarly to the current. The difference here is in the index notation, as opposed to the vector notation of Eq. 14. The indices here run $\mu = 0, 1, 2, 3$, whereas earlier only the spatial components $1 - 3$ were included. In general, 3-vectors will be presented with an arrow over them, and 4-vectors will have an index. Magnitudes of quantities will be shown as $|p|$ for a 4-vector (as in $p^\mu p_\mu$) and $|\vec{p}|$ or $\vec{p}^2$ for a 3-vector.

The interaction picture's interaction Hamiltonian thus becomes

$$H_{CI} = \int d^3x J^\mu(\vec{x}) A_\mu(\vec{x}) \tag{16}$$

where now, J^μ and A_μ are the 4-vector current and vector potential, respectively. Plugging in the current from Eq. 8 for a uniformly accelerated point charge, Eq. 16 becomes

$$\begin{aligned} H_{CI} &= \int d^3x \left(J_0(\vec{x})A_0(\vec{x}) - J_1(\vec{x})A_1(\vec{x})\right) \\ &= q \int d^3x \delta(x - \frac{1}{a}\cosh(a\tau))\delta(y)\delta(z) \left(A_0(\vec{x}) - \tanh(a\tau)A_1(\vec{x})\right) \\ &= qA_\mu(x(\tau), 0, 0)v^\mu \end{aligned} \tag{17}$$

where v^μ is as in Eq. 8 in Section 3.a. Thus the effect of this interacting Hamiltonian is to evaluate the vector potential at a certain point in space, namely, wherever the charge is.

3.d.2 Two-level Detector

This paper will often require a Minkowski observer located at a specific position for all time (trajectory $x^{\mu'} = (t, \vec{x})$), which will be modeled as a two-level system: basically, an atom with two possible states. The transition frequency between its ground state and excited state is taken to be ω_0. According to the Jaynes-Cummings model, the free Hamiltonian of the two-level system is of the form

$$H_{D_{free}} = \frac{\omega_0 \sigma_z}{2} \tag{18}$$

where the eigenstates are when the atom is excited ($|e\rangle$) and in its ground state ($|g\rangle$): σ_z, the inversion operator, is given by $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$. There is also the Hamiltonian of the free electromagnetic field:

$$H_{EM_{free}} = \sum_{k,s} \omega \left(a^\dagger_{k,s} a_{k,s} + \frac{1}{2} \right)$$

The operators $a^\dagger$ and a are the photon creation and annihilation operators, respectively (see [34, Chapter 14]). The free Hamiltonian is thus $H_0 = H_{D_{free}} + H_{EM_{free}}$.

The interaction of the two-level system with the electromagnetic field is modeled as either absorbing a photon and going from the ground to the excited state or emitting a photon and going from the excited to the ground state. The other terms are neglected via the rotating wave approximation (a valid approximation as the space between the walls of the “cavity” goes to infinity). We therefore introduce the operators $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$, which take the atom from the ground state to the excited state and the excited state to the ground state, respectively. Using the typical photon creation and annihilation operators $a^\dagger$ and a, along with a coupling strength given by a constant g, the interaction Hamiltonian is

$$H_{D1} = g(\sigma_+ a + \sigma_- a^\dagger) \tag{19}$$

However, the a here needs to be expressed in terms of the annihilation operators for each frequency: thus, integrating, we obtain:

$$a = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \epsilon_D^\mu \epsilon_{\mu r}^A a_{\mathbf{p}}^r$$

The rs again label a basis of polarization vector solutions (summed over): here, the A subscript indicates the polarization associated with the vector field (as found in Eq. 15) and ϵ_D^μ is the polarization of the detector. A similar approach can be taken with the creation operator. The nature of the calculation in Section 4 is such that the creation operator in Eq. 19 will always yield zero, and thus that term can be dropped and the annihilation operator can be converted into A_μ. Time dependence will be added to both σ and a similar to as in Section 3.d.1.

The total Hamiltonian is given by the sum of the free Hamiltonian and the interaction Hamiltonian:

$$H = H_0 + H_1 = H_{D_{free}} + H_{EM_{free}} + H_{D1}$$

4 Transition Amplitude Calculation

We now put together the elements of the background information illustrated above. The scenario takes place in Minkowski space: a lone observer sits still in a vacuum complete except for its own presence and that of a point charge. Said point charge, a classical fellow, has been accelerating with constant (proper) acceleration a for all of time, and will continue to do so in the future. Defining “seeing” as transitioning to the excited state, the question is simple: does the observer (a two-level system) initially in its ground state see the charge emit a photon?

If one were to take the easy way out, one could simply solve the equation $\Box A^\mu = J^\mu$ by employing Green’s function methods and be done with it. However, this paper seeks (in the spirit of quantum theory) to examine what happens when an actual detection is carried out: here, effects of the detector are included. This calculation will use second-order time dependent perturbation theory to determine the probability of the detector transitioning from the ground state to the excited state. Presumably, since the detector would remain in the ground state indefinitely if the charge were absent, such a detection would indicate that a uniformly accelerated charge does indeed radiate.

We are presently concerned with finding the overlap of evolved initial state ($U_I(t_f, t_0)\,|0, g\rangle$) with the state $|0, e\rangle$; that is, we want to find $\langle 0, e|U_I(t_f, t_0)|0, g\rangle$. U_I is given in Eq. 12 with the interaction Hamiltonians given by adding Eqs. 17 and 19: the interactions between the classical charge and the electromagnetic field and the two-level detector and the field, respectively. Since it is not possible to "turn off" the classical charge as it is the detector, we extend the integral over the charge's Hamiltonian to $\pm\infty$. The variable of integration, previously labeled t'' in Section 3.c, will be called τ as in Section 3.d.1 to highlight its role as proper time of the charge.

We can immediately discard all terms except the one involving the second-order correction, since taking the product with $|0, e\rangle$ and $|0, g\rangle$ would give zero for them. Higher odd order terms drop out similarly, while fourth- and higher even order corrections are smaller by factors of $\hbar$ and q (both small numbers) and can thus be neglected as even smaller than the small perturbation being considered. Another way to see this is that higher order terms are equivalent to the charge emitting a photon, the detector registering it, decaying, and the charge emitting another photon which the detector also registers. This will be less likely than just one photon being emitted and detected.

Hence we want to evaluate

$$
\begin{aligned}
\langle 0, e|U_I(t_f, t_0)|0, g\rangle &= \langle 0, e| - \frac{1}{2}\int_{t_0}^{t} dt'dt''T[H_I(t')H_I(t'')]|0, g\rangle \\
&= -\frac{1}{2}\,\langle 0, e| \int dt'dt''T[(H_{CI}(t') + H_{DI}(t'))(H_{CI}(t'') + H_{DI}(t''))]\,|0, g\rangle \\
&= -\frac{1}{2}\,\langle 0, e| \int dt'dt''H_{DI}(t')H_{CI}(t'')\,|0, g\rangle \qquad (20)
\end{aligned}
$$

The $H_{DI}(t')H_{DI}(t'')$, $H_{CI}(t')H_{DI}(t'')$, etc. terms are zero when sandwiched between $\langle 0, e|$ and $|0, g\rangle$ and are dropped in going from the second to the third line.

As mentioned at the end of Section 3.d.2, we can add terms that evaluate to zero. For example, the creation and annihilation operators in H_{DI} can be turned into the vector potential evaluated at the detector's 4-vector position $x^\mu = (t', l_x, l_y, l_z)$, the location of the detector. Creation of a photon will be canceled by taking the product with $\langle 0, e|$. The annihilation operators of photons that are not of the frequency of the produced photon give zero: the only remaining term will be the annihilation operator in Eq. 19. Thus, bringing in the explicit forms for H_{DI} and H_{C1} from Eqs. 17 and the modified version of Eq. 19, Eq. 20 now becomes:

$$\begin{aligned}\langle 0,e|U_I(t_f,t_0)|0,g\rangle &= -\frac{1}{2}\langle 0,e|\int_{t_0}^{t_f} dt' \int_{-\infty}^{\infty} d\tau e^{iH_0(t'-t_0)}\left(g\sigma_+\epsilon_D^\mu A_\mu(y^\mu)\right)\left(qA_\nu(y^\nu)v^\nu\right)|0,g\rangle \\ &= -\frac{qg}{2}\int_{t_0}^{t_f} dt' \int_{-\infty}^{\infty} d\tau e^{i\omega_0(t'-t_0)}\epsilon_D^\mu v^\nu \langle e|\sigma_+|g\rangle \langle 0|TA_\mu(x^\mu)A_\nu(y^\nu)|0\rangle \\ &= -\frac{qg}{2}\int_{t_0}^{t_f} dt' \int_{-\infty}^{\infty} d\tau e^{i\omega_0(t'-t_0)}\epsilon_D^\mu v^\nu D_P(x^\mu - y^\nu) \end{aligned} \tag{21}$$

The outer $e^{iH_0(t'-t_0)}$ term (from the interaction picture version of σ_+) acts on the $\langle 0,e|$ states to give $e^{i\omega_0(t'-t_0)}$. Remember that v^ν is a function of τ. Here, $y^\mu = \frac{1}{a}(\sinh(a\tau), \cosh(a\tau), 0, 0)$ is the trajectory of the accelerating charge. $D_P(x^\mu - y^\nu)$ is the photon propagator from Eq. 11. Converting to explicit form of the propagator, Eq. 21 becomes

$$\langle 0,e|U_I(t_f,t_0)|0,g\rangle = \frac{iqg}{2}\eta_{\mu\nu}\epsilon_D^\mu \int_{t_0}^{t_f} dt' \int_{-\infty}^{\infty} d\tau v^\nu e^{i\omega_0(t'-t_0)} \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq^\mu(x-y)_\mu}}{q^\mu q_\mu + i\epsilon}$$

Notating $q^\mu = (\omega, \vec{p})$, $d^\mu = x^\mu - y^\mu = (d^0, \vec{d})$ and generally using an arrow to denote a 3-vector, the expression simplifies to

$$\langle 0,e|U_I(t_f,t_0)|0,g\rangle = -\frac{iqg}{2}\eta_{\mu\nu}\epsilon_D^\mu \int_{t_0}^{t_f} dt' \int_{-\infty}^{\infty} d\tau \int \frac{d^3p}{(2\pi)^3}\int \frac{d\omega}{2\pi} v^\nu e^{i\omega_0(t'-t_0)} e^{-i\omega d^0}\frac{e^{i\vec{p}\cdot\vec{d}}}{\omega^2 - \vec{p}^2 + i\epsilon} \tag{22}$$

From here on, we drop the overall phase factor $e^{-i\omega_0 t_0}$.

The integral over p can be done by converting to polar coordinates, with $|\vec{p}| = r$ and $|\vec{d}| = d$:

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{d}}}{\vec{p}^2 - \omega^2 - i\epsilon} &= \frac{1}{(2\pi)^3}\int_0^\pi d\phi \int_{-1}^{1} d(\cos\theta) e^{ird\cos\theta}\int_0^\infty dr\, \frac{r}{2}\left(\frac{1}{r-(\omega+i\epsilon)} + \frac{1}{r+(\omega+i\epsilon)}\right) \\ &= \frac{i}{8\pi^2 d}\int_{-\infty}^{\infty} e^{ird}\left(\frac{1}{r-(\omega+i\epsilon)} + \frac{1}{r+(\omega+i\epsilon)}\right)\end{aligned}$$

after factoring $r^2 - \omega^2 - i\epsilon = (r-(\omega+i\epsilon))(r+(\omega+i\epsilon))$. Extending to the complex plane and closing a semi-circle contour in the upper half plane (such that the function decays as the radius of the contour goes to zero), the result is

$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{d}}}{\vec{p}^2 - \omega^2 - i\epsilon} = -\frac{1}{4\pi d}e^{i(\omega+i\epsilon)d} \tag{23}$$

We can also immediately evaluate the t' integral, since its only dependence is in two exponentials, and Eq. 22 becomes (with $\epsilon \to 0$):

$$\begin{aligned}\langle 0,e|U_I(t_f,t_0)|0,g\rangle = \frac{-iqg}{2}\eta_{\mu\nu}\epsilon_D^{\mu}\int_{-\infty}^{\infty} d\tau v^{\nu}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\frac{\omega}{a}\sinh(a\tau)} * -\frac{1}{4\pi d}e^{i\omega d} \\ * \frac{i}{\omega-\omega_0}\left(e^{-i(\omega-\omega_0)t_f} - e^{-i(\omega-\omega_0)t_0}\right) \end{aligned} \tag{24}$$

Calling $z = \sinh a\tau + ad$ ($d = |\vec{d}|$: for a detector located at the origin, $d = \frac{1}{a}\cosh(a\tau)$ and thus $z = e^{a\tau}$) and introducing a tiny offset in the ω_0 value, the ω integral can also be evaluated via contour integration. The denominator becomes $\omega - (\omega_0 + i\epsilon)$, thus chosen to have a positive imaginary part so that the pole is included when $z/a > t_f$ for the first exponential in Eq. 24 and when $z/a > t_0$ for the second. It makes sense to have the pole when this is the case rather than when $z/a < t_f$ or t_0 since z/a can be interpreted as the time of the detector.

The ω exponentials have residues of $e^{i(\omega_0+i\epsilon)(z/a-t)}e^{i\omega_0 t}$ and $e^{i(\omega_0+i\epsilon)(e^{a\tau}/a-t_0)}e^{i\omega_0 t_0}$ which are both $e^{i\frac{\omega_0}{a}z}$. With this, Eq. 24 simplifies:

$$\begin{aligned}\langle 0,e|U_I(t_f,t_0)|0,g\rangle &= \frac{-qg}{16\pi^2}\eta_{\mu\nu}\epsilon_D^{\mu}\int_{-\infty}^{\infty}\frac{d\tau}{d}v^{\nu} * 2\pi i e^{i\frac{\omega_0}{a}z}\left[\theta\left(\frac{z}{a}-t_f\right)-\theta\left(\frac{z}{a}-t_0\right)\right] \\ &= \frac{iqg}{8\pi}\eta_{\mu\nu}\epsilon_D^{\mu}\int_{z_1}^{z_2}\frac{d\tau}{d}v^{\nu}e^{i\frac{\omega_0}{a}z}\end{aligned} \tag{25}$$

where $z_1 = \frac{t_0}{a}$ and $z_2 = \frac{t_f}{a}$. This step has also taken $\epsilon \to 0$.

Combining the Heaviside step functions in going from the first to the second line of Eq. 25 is demonstrated in Figure 3a. Their effect is to cut short the limits of the τ integral, as seen in the second line of Eq. 25. Note that if t_0 is negative, then for a detector at $x = 0$ and any y, z, the lower limit stays at zero, since the minimum value of d is $\frac{1}{a}\cosh(a\tau)$ and thus $z = \frac{1}{a}\sinh(a\tau) + d$ only runs over positive values. For a detector displaced along the x-axis at position l_x, it is possible to achieve negative values when $\cosh(a\tau) < l_x$.

To make further progress we specialize to the case where the detector is located at a position $\vec{d} = (0, l_y, l_z)$ (the x-position is still zero because otherwise it becomes quite monstrous to solve even d). Expressing a variety of functions in terms of z yields:

$$d = \frac{1}{a}(\cosh^2(a\tau) + a^2 l^2)^{1/2} = \frac{1}{2az}\left(4z^2b^2 + (z^2-b^2)^2\right)^{1/2} \tag{26}$$

$$\frac{dz}{d\tau} = \frac{az\left(4z^2 + (z^2-b^2)^2\right)^{1/2}}{z^2+b^2} \tag{27}$$

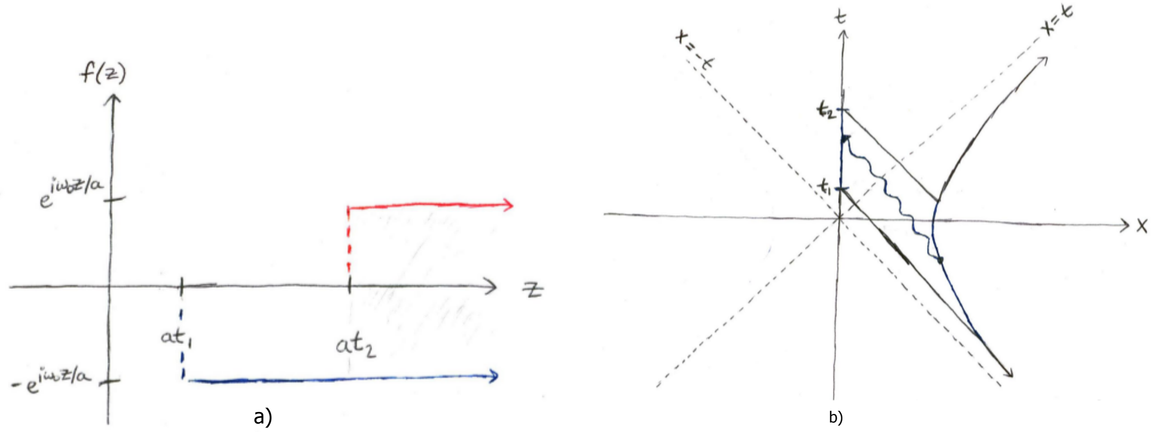

Figure 3: a) The Heaviside step functions in Eq. 25 simplify nicely: the two functions are zero before $z = at_1$ and cancel each other out past $z = at_2$. The only region the function is nonzero is when $at_1 < z < at_2$. b) The restrictions on z, the time of the charge as seen by the detector, are shown by the 45° lines. A photon emitted within this time period will reach a detector sitting at the origin, whereas photons emitted outside of this time period will not.

where $l^2 = l_y^2 + l_z^2$ and $b^2 = 1 + a^2 l^2$. Note that bc^2/a is the distance of the detector from the charge at $t = 0$ (when the charge is instantaneously at rest).

Substituting Eqs. 26 and 27 into Eq. 25 and expanding the product $\eta_{\mu\nu}\epsilon_D^{\mu}v^{\nu} = \epsilon^0 - \epsilon^1 \tanh(a\tau)$ for $\epsilon^0 = 0$ yields a formula only in terms of z:

$$\langle 0, e|U_I(t_f, t_0)|0, g\rangle = \frac{-iqg}{4\pi}\epsilon^1 \int_{z_1}^{z_2} \frac{z^4 - b^4}{(4z^2 + (z^2 - b^2)^2)^{3/2}} e^{i\frac{\omega_0}{a}z} dz \tag{28}$$

For the case $l = 0$, which corresponds to a detector at the origin and $b^2 = 1$, the integral simplifies to

$$\langle 0, e|U_I(t_f, t_0)|0, g\rangle = \frac{-iqg}{4\pi}\epsilon^1 \int_{z_1}^{z_2} \frac{z^2 - 1}{(z^2 + 1)^2} e^{i\frac{\omega_0}{a}z} dz \tag{29}$$

This result is interesting because, even though the initial problem was posed such that the τ integral ran over all of time, the time limits were cut down anyway by virtue of the connection between absorption and emission. Figure 3b shows how the finite speed of light corresponds to a limitation on when the charge emitted a photon. Equation 29 means that it doesn't matter what trajectory the charge takes when the detector is off; it will not affect the probability that the detector detects a photon. The action of the charge when the detector was off was originally thought to influence the detection probability. Since it does not, perhaps the presence of radiation could actually be tested experimentally via a linear accelerator, although care would have to be taken not to accidentally excite the detector or detect radiation from when the charge starts/stops accelerating.

The integral in Eq. 29 cannot easily be evaluated further, and now the task is to investigate interesting properties of this integral. For example, what is the most probable value of ω? How does the probability amplitude depend on the acceleration a? Exploring this integral further via numeric integration is the subject of the next section.

5 Transition Probability and Its Dependencies

The final result found in Section 4's Equation 29 depends on five variables: the acceleration a of the point charge, the transition frequency ω_0 of the detector, the position l of the detector, and the starting and ending times. In the sections below we explore all five of these parameters. The pre-factor $\frac{-iqg}{4\pi}\epsilon^1$ of quantities with units in Eq. 29 sets the scaling of the final result, so the dimensionless integrals were arbitrarily scaled to their maximum value. Note that, although the detector's polarization vector could have components along the y- and z-axes, these components would be zeroed out by v^ν.

5.a Infinite Time

In this section, Eq. 29 is numerically integrated and plotted separately for a range of acceleration and transition frequency values. The integrals are evaluated from $t_1 = 0$ or close to zero up to a time after which the distributions changed only imperceptibly, effectively taking $t_f \to \infty$. Different values of $l \geq 0$ are also plotted.

From Figure 4, it is reassuring that the probability of detecting a photon is zero when the charge and detector are at rest with respect to each other. This fact comes from the factor of a outside the integral in the τ version of Equation 29. This figure is interesting because the transition probability actually peaks. This may be rationalized because, as the acceleration increases, the time limits get closer and closer until they are approximately equal (both go to zero): that is, for a detector at the origin ($z = e^{a\tau}$),

$$\lim_{a\to\infty} \frac{\ln(at_f)}{a} - \frac{\ln(at_0)}{a} \to 0$$

Since the function is not constant between $a = 0$ and $a \to \infty$, it must have a maximum somewhere (not a minimum, since the absolute value is by definition positive). As previously noted, the case $l = 0$ is a strange location. The probability decays sharper than expected and then increases again, decreasing again for higher values of acceleration than shown in Figure 4. The reason for the function's two maxima would be interesting to explore.

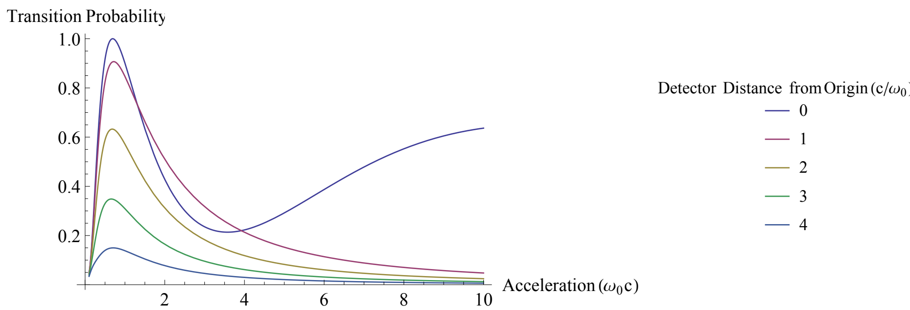

Figure 4: Transition Probability as a function of charge acceleration. The upper and lower time limits are $t_0 = 0.1/\omega_0$ and $t_f = 50/\omega_0$. The case $l = 0$ constitutes a detector at the origin, a location known to be a special case. This case not included, the transition probability decreases the further away the detector is from the origin, and the acceleration of the peak transition probability shifts to smaller accelerations.

The probability of detecting a photon depends as expected on the transition frequency of the detector, as shown in Figure 5. If the transition frequency is zero, the system basically has only one state, the ground state, and thus is never excited. The probability of detection goes down as the detector is placed further away from the origin, which makes sense because that is where the radiation is concentrated. The frequency at which detection is most probable changes with distance: this is because the first photons to reach a detector further away from the origin were emitted later (when the charge was moving slightly less quickly towards the origin) and thus redshifted as compared to the $l = 0$ photons, resulting in a lower frequency to be detected by the detector. There seems to be no strange behavior at the origin in this figure.

It would be extremely interesting to move the detector in the x-direction and discover what happens when the detector is located at $x = 1/a$ and greater and lesser values.

5.b Finite Time

This section takes typical values of the acceleration and detector transition frequency and explores how Eq. 29 depends on the times that the detector is turned on and off. Remember that the detector will not see anything before $t_1 = 0$, since the radiation from the charge in the infinite past will only reach the origin at $t = 0$.

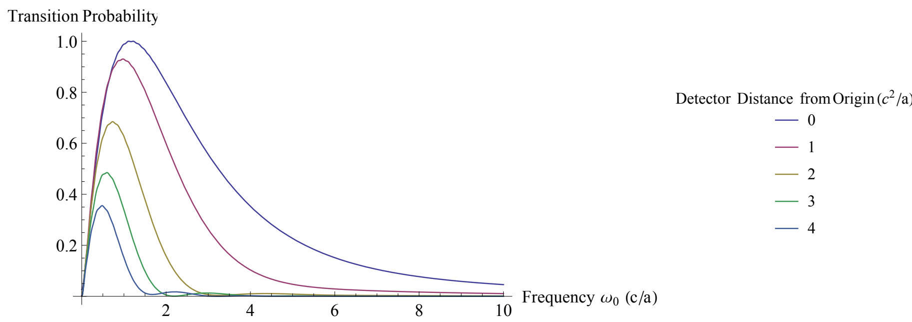

Figure 5: Transition probability as a function of detector transmission frequency. The upper and lower time limits were taken as $t_0 = 0$ and $t_f = 50a/c$. The frequency that is most likely to detect radiation shifts to lower frequencies further away from the origin.

Figure 6 shows the result of changing what time the detector was turned off (fixing the time it was turned on at approximately zero). It makes sense that the probability of transition frequency should increase the longer the detector is left on and indeed this is confirmed in Figure 6. However, the increase is not monotonic. Also, the probability decreases as the distance from the origin increases, which makes sense because radiation emitted in the far past would never reach a detector dislocated from the origin.

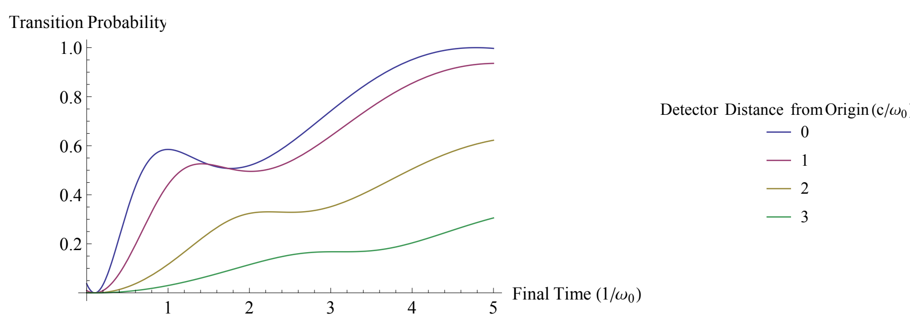

Figure 6: Transition probability as a function of when the detector is turned off. The detector is turned on at $t_1 = 0$.

Figure 7 holds the time the detector is turned off at essentially infinity and varies the time the detector was turned on from zero up to the time it is turned off ($t = 5/\omega_0$ on the plot axis). For initial times close to final times, the probability of transition is close to zero, which makes sense because there is a smaller (or non-existent) time window in which a photon must be emitted by the charge in order to be detected. The dependency on detector displacement from the origin is also rather unexpected; we would not have expected the lines of different detector distance to cross.

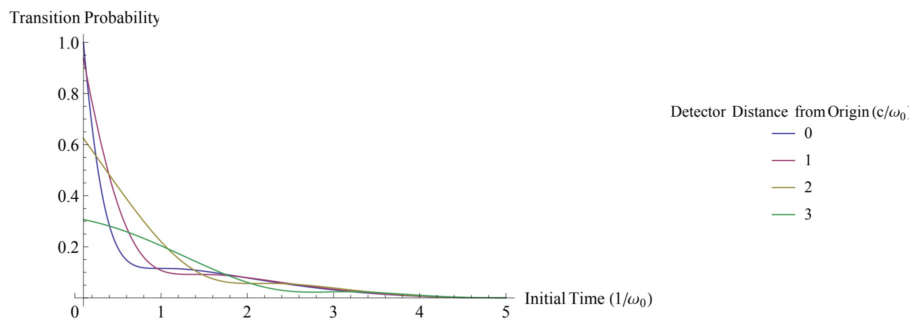

Figure 7: Transition probability as a function of when the detector is turned on. The detector is turned off at $t_2 = 5/\omega_0$.

6 Conclusions

Clearly, this paper's method of calculating radiation yields a nonzero result and hence an affirmative answer to the question of whether a uniformly accelerating charge radiates or not. It should be noted that this calculation studied only one of many possible situations involving a charge and a detector: a detector stationary in Minkowski space observing a Rindler charge. We could have also looked at a charge supported in a uniform gravitational field as seen by a freely-falling observer (equivalent to this paper's situation according to the equivalence principle) or a similarly supported observer (equivalent to a co-accelerating charge and detector).

There is another parameter to vary here; namely, whose vacuum are we defining. This paper uses a Minkowski vacuum, but we could have just as well looked at the so-called Boulware vacuum. For visualization purposes, assume the charge is placed in a box. The box is accelerated so the charge sees a thermal bath of particles due to the Unruh effect and the box is then evacuated. It is now a vacuum from the accelerating charge's perspective: this is the Boulware vacuum. This is what a vacuum on Earth is. In a sort of reverse Unruh effect, the Minkowski observer will then see a thermal bath of particles because the Minkowski vacuum can be expressed in terms of the Rindler creation and annihilation operators as

$$|0\rangle_M = \prod_j \left[\sqrt{1 - e^{-2\pi\omega_j/a}} \sum_{n_j} e^{-\pi n_j \omega_j/a} \left|n_j\right\rangle_L \left|n_j\right\rangle_R \right] \tag{30}$$

where $|n\rangle_L$ and $|n\rangle_R$ are the states with n photons in the left and right wedges (regions I and III), respectively [36]: the number of photons in either wedge is the same. Here, the Unruh temperature $kT = \hbar a/2\pi$ has been substituted in. The weighting factor gives the probability that a photon of frequency ω will be produced. Similarly, the Rindler (or Boulware) vacuum constitutes a superposition of Minkowski creation and annihilation operators.

Considering the situation from both perspectives yields an interpretation that explains the presence of radiation while not violating the equivalence principle. From the charge's point of view, it is at rest in a gravitational field and sees a thermal bath of particles. Pair production in this thermal bath will result in both the charge and the detector absorbing a photon, similar to as in Unruh and Wald's paper, but with an actual detector judging the presence of radiation instead of using the overall state as an indicator [27]. From the detector's perspective, the charge is accelerating in a vacuum. It emits a photon, which is then absorbed by the detector. These two situations are illustrated in Figure 8.

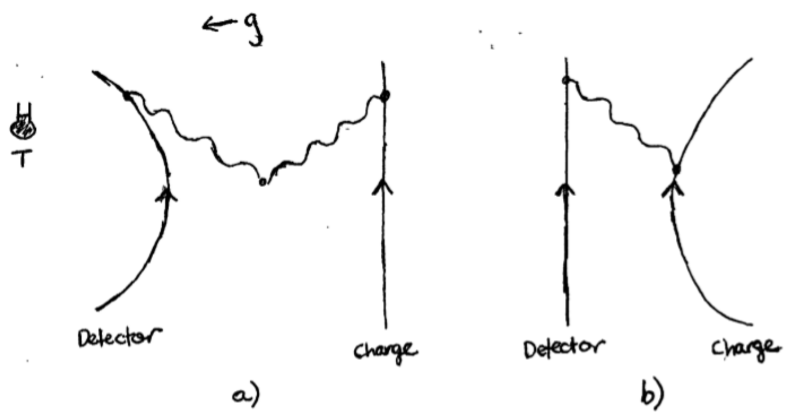

Figure 8: A comparison of the two viewpoints. a) From the charge's perspective, it is sitting in a gravitational field and the detector is freely falling. Two photons are pair-created from the thermal bath it sees and one is detected by the detector. b) From the detector's perspective, the charge is accelerating and emits a photon.

Using this new pespective due to the Unruh effect, we can also begin to justify Boulware's claims that radiation from a charge is undetectable by a co-accelerating observer. In this situation, both the charge and the detector would see a thermal bath of particles due to the Unruh effect. Boulware had claimed that it would be impossible for the detector to tell if radiation was due to the presence of the charge or not: indeed, if the detector is accelerating, it will see a thermal bath regardless of whether the charge is there or not. Boulware's view can be interpreted as in Figure 9, with one photon being absorbed by either the detector or the charge and the other photon disappearing behind the horizon. In this way, the Unruh effect validates the equivalence principle and resolves the paradox posed in Section 2.c.

It is this fundamental inequality of the vacuum state as seen by two different observers that seems to have fostered so much misunderstanding in previous papers. Indeed, those that thought a charge would not radiate were likely thinking of the Boulware vacuum: in this case it might seem that the situation is static (stationary with time-reversal symmetry). However, this staticity is only from the charge's viewpoint and a Minkowski observer would, as already mentioned, see a bath of particles. The situation is therefore not static from a stationary observer and thus this line of reasoning does not hold.

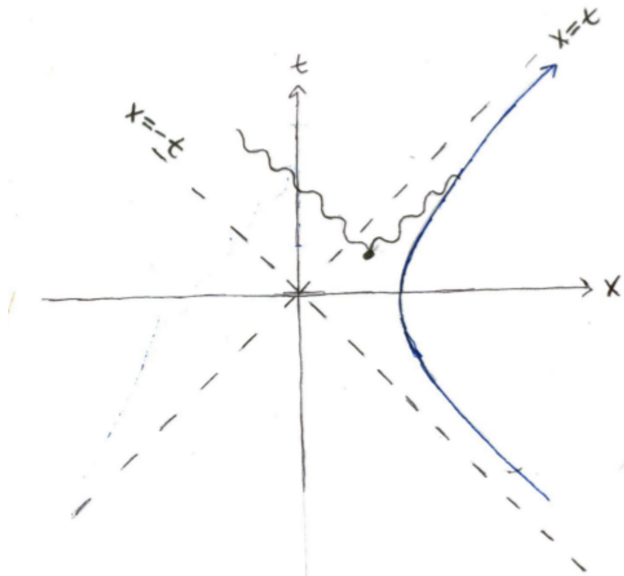

Figure 9: When the detector and charge are co-accelerating as indicated by the blue hyperbolic path, pair creation in the thermal bath results in detection of only one photon. The other propagates to the left and falls behind the horizon.

There are still mysteries and subtleties to work out about the calculation that resulted in Eq. 29 beyond the equivalence principle. For example, what would it mean for the detector polarization to have a zeroth time component? This would not be zeroed out by v^ν and (if it is physically reasonable) could be interesting. In addition, this paper's semi-classical treatment could be extended to a fully quantum mechanical one. This would involve treating the charge quantum mechanically, with uncertainty in position and more questions on how it would be created and accelerated. More thought could be given to the mechanism of accelerating the charge; for example, if it is via an electric field, perhaps the Schwinger effect would become important. Finally, it would be worth exploring the results of Section 5 and moving the detector along the x-axis.

As a last note, many people regard the Unruh effect as a correction to the classical radiation rate. For McDonald, Unruh radiation means that classical radiation exists [37]. However, it is tricky to reconcile these views with this paper's since this paper does not calculate the Unruh radiation directly and they are connected only through the ideas of the vacuum. McDonald also mentions that the previous papers discussed in Section 2 had been unintentionally anticipating Hawking radiation as a resolution to the equivalence principle, which is the view the present paper holds as well [37].

The question of radiation and uniformly accelerated charges has a long history of attracting many physicists due to its very fundamental nature. This paper has provided a new perspective on the ongoing debate and, through the Unruh effect, explains why some of the past literature had been led astray. Though the question is perhaps not conclusively resolved, it has at least been clarified and the deep connection between the problem and the importance of defining the vacuum properly has been revealed.

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This paper represents my own work in accordance with University regulations.

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